THE VARIANCE OF A RANK ESTIMATOR OF TRANSFORMATION MODELS

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This note shows that the asymptotic variance of Chen's [Econometrica, 70, 4 (2002), 1683–1697] two-step estimator of the link function in a linear transformation model depends on the first-step estimator of the index coefficients.

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1 The model and the estimator

For an unspecified strictly-increasing function $\Lambda_0(\cdot) : \mathscr{R} \mapsto \mathscr{R}$, the linear transformation model takes the form

$$\Lambda_0(Y) = X\beta + \varepsilon,$$

where ε is a latent disturbance, distributed independently of the covariates X, and β is an unknown coefficient vector of conformable dimension. Set $\Lambda_0(y_0) = 0$ for a chosen baseline value y_0 and assume that $\beta = (1, \alpha_0')'$.

Let $W_i \equiv (Y_i, X_i)$ $(i = 1, \ldots, n)$ be observations on $W \equiv (Y, X)$, drawn at random from a distribution P that is supported on a set $\mathscr{W}$. Let $b_n \equiv (1, \alpha_n')'$ be a first-step estimator of β. For fixed y, Chen (2002) proposed estimating $\Lambda_0 \equiv \Lambda_0(y)$ by Λ_n, which maximizes

$$\frac{1}{n(n-1)} \sum_{i \neq j} h(W_i, W_j, y, \Lambda, b_n), \quad h(W_1, W_2, y, \Lambda, b) \equiv [1\{Y_1 \geq y\} - 1\{Y_2 \geq y_0\}]\ 1\{(X_1 - X_2)b \geq \Lambda\},$$

with respect to Λ over a compact subset of the real line containing Λ_0.

2 The influence function

Impose Assumptions 1–5 of Chen (2002). Strenghten Assumption 6 by demanding α_n to be asymptotically linear, that is, $\sqrt{n}(\alpha_n - \alpha_0) = n^{-1/2} \sum_i \psi(W_i) + o_p(1)$ for a function $\psi(\cdot)$ that has zero mean and finite variance under P. Fix y throughout and leave the dependence of quantities on it implicit unless there is risk of confusion. Let $\tau(w, \Lambda, b) \equiv Eh(W, w, y, \Lambda, b) + Eh(w, W, y, \Lambda, b)$, $V \equiv \frac{1}{2} E\nabla_{\Lambda\Lambda}\tau(W, \Lambda_0, \beta)$, and $\Omega \equiv E\nabla_{\Lambda\alpha'}\tau(W, \Lambda_0, \beta)$. Arguments along the lines of those in Sherman (1993) and Asparouhova *et al.* (2002) yield $\sqrt{n}(\Lambda_n - \Lambda_0) = n^{-1/2} \sum_i I(W_i) + o_p(1) \xrightarrow{\mathcal{L}} \mathcal{N}(0, EI(W)^2)$ for

$$I(w) \equiv -V^{-1}\left[\nabla_\Lambda \tau(w, \Lambda_0, \beta) + \frac{1}{2}\Omega\ \psi(w)\right] = J(w) - \frac{1}{2}V^{-1}\Omega\ \psi(w).$$

Chen (2002, pp. 1687 and Theorem 1) argues that $\Omega = 0$, so that $I(w) = J(w)$ and the asymptotic variance of $\sqrt{n}(\Lambda_n - \Lambda_0)$ is unaffected by the estimation noise in b_n.

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Write f and p_z for the densities of ε and $Z \equiv X\beta$, respectively. Arrange the components of $X = (X_1, \widetilde{X}) \in \mathscr{R} \times \mathscr{X}$ so that the distribution of scalar X_1 given $\widetilde{X} = \widetilde{x}$ satisfies the absolute-continuity requirement of Chen (2002, Assumption 2) for all $\widetilde{x}$ in $\mathscr{X}$. The calculations summarized below show that, with $\mathcal{X}(z) \equiv E[\widetilde{X}|Z = z]$,

$$-V = \int_{-\infty}^{+\infty} f(-z)\ p_z(z + \Lambda_0)\ p_z(z)\ \mathrm{d}z, \tag{1}$$

$$\frac{1}{2}\Omega = \int_{-\infty}^{+\infty} f(-z)\ p_z(z + \Lambda_0)\ p_z(z)\big[\mathcal{X}(z + \Lambda_0) - \mathcal{X}(z)\big]\ \mathrm{d}z. \tag{2}$$

Equation (2) reveals that Ω will generally be non-zero. Like V, it can be estimated by the cross-derivative of a smoothed version of the symmetrized objective function evaluated at (Λ_n, b_n). Consistency follows under conditions analogous to those for the estimator of V stated in Chen (2002, pp. 1695–1696).

The conclusions drawn here extend to the case where the observations on Y are subject to random censoring. The appropriate modification to the influence function stated in Chen (2002, Theorem 2) is readily derived.

3 Calculations

Let $\tau(w) = \tau(w, \Lambda, b)$ for fixed values Λ and $b = (1, \alpha')'$. Write $\tau_0(w)$ for $\tau(w, \Lambda_0, \beta)$, $\nabla_\Lambda \tau_0(w)$ for $\nabla_\Lambda \tau(w, \Lambda_0, \beta)$, etc. Manipulate the inequalities in $\tau(w)$ to see that

$$\begin{aligned}\tau(W) = &\int_{\mathscr{W}} (1\{y < y_0\} - 1\{Y < y\})\ 1\{xb \le Xb - \Lambda\}\ \mathrm{d}P(w) \\ &- \int_{\mathscr{W}} (1\{Y < y_0\} - 1\{y < y\})\ 1\{xb < Xb + \Lambda\}\ \mathrm{d}P(w) + c_0\end{aligned}$$

for $c_0 \equiv \int (1\{Y < y_0\} - 1\{y < y\})\ \mathrm{d}P(y)$, which does not depend on (Λ, b). Let $p_z(z|\widetilde{x})$ be the density of Z given $\widetilde{X} = \widetilde{x}$ at z and let $\Delta_\alpha(\widetilde{X}, \widetilde{x}) \equiv Z + (\widetilde{X} - \widetilde{x})(\alpha - \alpha_0)$. By iterated expectations,

$$\tau(W) = -\int_{\mathscr{X}} \int_{-\infty}^{\Delta_\alpha(\widetilde{X},\widetilde{x})-\Lambda} S_{y,y_0}(Y, z)\ p_z(z|\widetilde{x})\ \mathrm{d}z\ \mathrm{d}P(\widetilde{x}) + \int_{\mathscr{X}} \int_{-\infty}^{\Delta_\alpha(\widetilde{X},\widetilde{x})+\Lambda} S_{y_0,y}(Y, z)\ p_z(z|\widetilde{x})\ \mathrm{d}z\ \mathrm{d}P(\widetilde{x}) + c_0,$$

where $S_{y_1,y_2}(Y, Z) \equiv 1\{Y < y_1\} - F(\Lambda_0(y_2) - Z)$ and $F(z) \equiv \int_{-\infty}^{z} f(z)\ \mathrm{d}z$.

Use Leibniz's rule to verify that

$$\begin{aligned}\nabla_\Lambda \tau(W) = &\int_{\mathscr{X}} S_{y,y_0}(Y, \Delta_\alpha(\widetilde{X}, \widetilde{x}) - \Lambda)\ p_z(\Delta_\alpha(\widetilde{X}, \widetilde{x}) - \Lambda|\widetilde{x})\ \mathrm{d}P(\widetilde{x}) \\ &- \int_{\mathscr{X}} S_{y_0,y}(Y, \Delta_\alpha(\widetilde{X}, \widetilde{x}) + \Lambda)\ \ p_z(\Delta_\alpha(\widetilde{X}, \widetilde{x}) + \Lambda|\widetilde{x})\ \mathrm{d}P(\widetilde{x})\end{aligned} \tag{3}$$

and that $\nabla_\Lambda \tau_0(W) = S_{y,y}(Y, Z)\ p_z(Z - \Lambda) - S_{y_0,y_0}(Y, Z)\ p_z(Z + \Lambda)$. Notice that $E\nabla_\Lambda \tau_0(W) = 0$ because

$$E[S_{y,y}(Y, Z)|Z = z] = 0 \tag{4}$$

for any y.

Differentiate with respect to Λ under the integral sign in Equation (3), re-arrange, and evaluate at (Λ_0, β) to obtain

$$\nabla_{\Lambda\Lambda}\tau_0(W) = -f(\Lambda_0 - Z)\ p_z(Z - \Lambda_0) - f(-Z)\ p_z(Z + \Lambda_0) - c_1$$

where $c_1 \equiv S_{y,y}(Y,Z)\ p_z'(Z - \Lambda_0) + S_{y_0,y_0}(Y,Z)\ p_z'(Z + \Lambda_0)$ and p_z' is the derivative of p_z. Integrate and apply the moment condition in Equation (4) to dispense with c_1. Then

$$2V = -\int_{-\infty}^{+\infty} f(\Lambda_0 - z)\ p_z(z - \Lambda_0)\ p_z(z)\ \mathrm{d}z - \int_{-\infty}^{+\infty} f(-z)\ p_z(z + \Lambda_0)\ p_z(z)\ \mathrm{d}z.$$

Equation (1) follows on a change of variable from z to $z - \Lambda_0$ in the first integral.

Follow the same steps to deduce that

$$\nabla_{\Lambda\alpha'}\tau_0(W) = f(\Lambda_0 - Z)\ p_z(Z - \Lambda_0)\ [\widetilde{X} - \mathcal{X}(Z - \Lambda_0)] - f(-Z)\ p_z(Z + \Lambda_0)\ [\widetilde{X} - \mathcal{X}(Z + \Lambda_0)] + c_2$$

for $c_2 \equiv S_{y,y}(Y,Z)\ p_z'(Z - \Lambda_0)[\widetilde{X} - \mathcal{X}(Z - \Lambda_0)] - S_{y_0,y_0}(Y,Z)\ p_z'(Z + \Lambda_0)\ [\widetilde{X} - \mathcal{X}(Z + \Lambda_0)]$. Because c_2 has zero mean by the condition in Equation (4),

$$\Omega = \int_{-\infty}^{+\infty} \Big\{ f(\Lambda_0 - z)\ p_z(z - \Lambda_0)\ \big[\mathcal{X}(z) - \mathcal{X}(z - \Lambda_0)\big] - f(-z)\ p_z(z + \Lambda_0)\ \big[\mathcal{X}(z) - \mathcal{X}(z + \Lambda_0)\big] \Big\}\ p_z(z)\ \mathrm{d}z.$$

A change of variable then establishes Equation (2).

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